Algebraic Number Theory
Mid-sem. test, BMath-3rd Year

INSTRUCTIONS: Solve any **four** problems. Maximum score is 20. Time: 3 hours. You may use results proved in the class without proving them.

1. Let I be the ideal $(2, 1+\sqrt{-3})$ in the ring $\mathbb{Z}[\sqrt{-3}]$.
   (i) Show that $I \neq (2)$ and $I^2 = 2I$.
   (ii) Show that $\mathbb{Z}[\sqrt{-3}]$ is not a Dedekind domain. (3+2)

2. Let $K = \mathbb{Q}(\sqrt{6})$. Let I be the ideal $I = (2, \sqrt{6}) \subset R$ in the ring of integers $R = \mathbb{Z}[\sqrt{6}]$. Prove that $I^2 = (2)$ and that $||I|| = 2$. (5)

3. In the number ring $\mathbb{Z}[\sqrt{-5}]$ consider the ideal $I = (1+\sqrt{-5}, 3+\sqrt{-5}, 19+9\sqrt{-5})$. Determine explicit $\alpha, \beta \in I$ such that $I = (\alpha, \beta)$. (5)

4. Let K be a number field, $\mathcal{O}_K$ its ring of integers and P a nonzero prime ideal in $\mathcal{O}_K$. Let $a \in \mathcal{O}_K$ be such that $P \nmid (a)$. Prove that $a^{||P||-1} = 1 \mod P$. (5)

5. Let K be a number field and I a non-zero ideal in the ring of integers of K. Prove that $||I|| \in I$. (5)

6. Let K be a number field and I and J be nonzero ideals in its ring of integers. Let $I = P_1^{r_1} \cdots P_t^{r_t}$ and $J = P_1^{s_1} \cdots P_t^{s_t}$, where the ideals P_i are prime ideals of $\mathcal{O}_K$ and r_i and s_i are nonnegative integers. Find prime factorizations for $gcd(I, J)$ and $lcm(I, J)$. (5)